B.Sc. Part I

2nd Paper

Integral calculus

Reduction formula

I. Evaluate $\int_0^{\pi/2} \sin^n x \, dx$.

Soln. Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$ ——— (1)

$\Rightarrow I_n = \int_0^{\pi/2} \underbrace{\sin^{n-1} x}_{U} \cdot \underbrace{\sin x}_{V} \, dx$

$= \left[\sin^{n-1} x \int \sin x \, dx\right]_0^{\pi/2} - \int_0^{\pi/2} \left[\frac{d}{dx}(\sin^{n-1} x) \int \sin x \, dx\right] dx$

$= -\left[\sin^{n-1} x \cdot \cos x\right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cdot \cos x \cos x \, dx$

$= 0 + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cdot \cos^2 x \, dx$

$\Rightarrow I_n = (n-1) \int_0^{\pi/2} \sin^{n-2} x (1 - \sin^2 x) \, dx$

$\Rightarrow I_n = (n-1) \int_0^{\pi/2} \sin^{n-2} x \, dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx$

$\Rightarrow I_n = (n-1) I_{n-2} - (n-1) I_n$

or $I_n(1+n-1) = (n-1) I_{n-2}$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} \quad \text{——— (2)}$$

Replacing n by $n-2$, ~~$n-4$, $r-r$~~

$$\Rightarrow I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

So, eq. (2) becomes
$$I_n = \frac{n-1}{n} \frac{n-3}{n-2} I_{n-4}$$

Similarly
$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6}$$

$$\Rightarrow I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} I_{n-8} \quad \text{— (3)}$$

<u>Case I</u> If n is even then

$$\Rightarrow I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} I_0 \quad \text{— (4)}$$

$$\because I_n = \int_0^{\pi/2} \sin^n x \, dx$$

$$\Rightarrow I_0 = \int_0^{\pi/2} (\sin x)^0 \, dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

Putting this value in (4), we get

$$\boxed{I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}}$$

(n is even)

Case II If n is odd Then

from eq (3), we've

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} I_1 \quad \text{—(5)}$$

from (1) we've $I_n = \int_0^{\pi/2} \sin^n x \, dx$

Put $n = 1$

$$\Rightarrow I_1 = \int_0^{\pi/2} \sin x \, dx = -\left[\cos x\right]_0^{\pi/2}$$

$$\Rightarrow I_1 = -\left(\cos\frac{\pi}{2} - \cos 0\right) = 1.$$

Putting this value in (5) we get

$$\boxed{I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}}$$

(when n is odd.)